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Super integrable four-dimensional autonomous mappings

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Abstract

A systematic investigation of the complete integrability of a fourth-order autonomous difference equation of the type $w(n+4) = w(n)F(w(n+1), w(n+2), w(n+3))$ is presented. We identify seven distinct families of four-dimensional mappings which are super integrable and have three (independent) integrals via a duality relation as introduced in a recent paper by Quispel, Capel and Roberts (2005 *J. Phys. A: Math. Gen.* **38** 3965–80). It is observed that these seven families can be related to the four-dimensional symplectic mappings with two integrals including all the four-dimensional periodic reductions of the integrable double-discrete modified Korteweg–deVries and sine-Gordon equations treated in an earlier paper by two of us (Capel and Sahadevan 2001 *Physica* A **289** 86–106).

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1. Introduction

Substantial progress in the theory of integrable mappings was made by the introduction of the 18-parameter Quispel, Roberts and Thompson (QRT) family of two-dimensional mappings [1, 2] obtained by differencing an integral $I(w, x)$ with $w = w(n), w(n+1) = x(n) = x$, that is a rational biquadratic function of w and x. The iterated values $w(n+1) = w'$ and $x(n+1) = x'$ are defined by the following relation:

$$I(w, x) - I(w', x') = 0.$$

Higher dimensional integrable mappings could be obtained by reduction from known integrable difference equations; see, e.g., [3–5] and also [6] for a general review. In [7], we studied a family of four-dimensional mappings arising from an integral $I(w, x, y, z), w = w(n), x = x(n), y = y(n), z = z(n)$ which is a rational function with the numerator and denominator being quadratic polynomials in terms of the variables (w, x, y, z). By taking the difference $I(w, x, y, z) - I(w', x', y', z') = 0$ one can obtain a 162-parameter family of mappings which is not integrable in general. However in [7] we have considered the special case that the integral $I(w, x, y, z)$ is invariant under cyclic permutation of coordinates, that is

$$I(w, x, y, z) = I(x, y, z, w) = I(y, z, w, x) = I(z, w, x, y) \tag{1.1}$$

and we have restricted ourselves to mappings having the symplectic structure with (antisymmetric) 2-form

$$\Omega(\xi, \eta) = C_{\xi\eta}\xi\eta, \qquad (\xi, \eta = w, x, y, z) \tag{1.2}$$

with $C_{\xi\eta}$ being constants that are invariant under iterations of the mapping. As a consequence, apart from several cases that can be reduced to two-dimensional mappings of the QRT family, we have identified six families of four-dimensional symplectic mappings of type G

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = \frac{1}{w}G(x, y, z), \tag{1.3}$$

with explicit expressions for $G(x, y, z)$. The reported six families can also be identified with the four-dimensional reductions arising from periodic solutions

$$u(n + z_2, m - z_1) = u(n, m) \tag{1.4}$$

of the integrable discrete–discrete modified Korteweg–de Vries ($\Delta\Delta$mKdV) and integrable discrete–discrete sine-Gordon ($\Delta\Delta$s-G) on the two-dimensional lattice $(m, n) \in \mathbb{Z}_2$ of [4]. From this it follows, cf [4], that six families of symplectic mappings admit a second independent integral $K(w, x, y, z)$ which can be constructed from the trace of monodromy matrices formed by products of Lax matrices [4] and which is not cyclic invariant in general. Hence the identified six families of four-dimensional symplectic mappings are completely integrable in the sense of Liouville [6, 8].

In this paper, we consider four-dimensional mappings of type F having the form

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = wF(x, y, z) \tag{1.5}$$

and derive seven distinct families of super integrable mappings, that is they are measure preserving and admit three independent integrals of motion. Taking a general linear combination of the constructed three integrals we can consider a duality relation in the sense of [9], that is

$$I(w, x, y, z) - I(x, y, z, z') = f(w, x, y, z, z')f^*(w, x, y, z, z'), \tag{1.6}$$

in which $f = 0$ is one of the seven integrable equations of type F. Considering the dual equations $f^* = 0$ it turns out that they can be identified as one of the six symplectic and integrable mappings of type G in (1.3) given in [7], apart from some more mappings that can be reduced to two-dimensional mappings of the QRT family [2]. (From a more general expression for $I(w, x, y, z)$ including another (dependent) integral as well we obtain in a number of cases a dual mapping with two integrals but with no obvious symplectic structure, in complete analogy with the example in equation (32) of [9]).

To investigate the super integrable mappings of type F we consider an integral $I(w, x, y, z)$ having the form

$$\begin{aligned} I(w, x, y, z) = \frac{1}{wxyz}[&A_{11}(x, y)w^2z^2 + A_{12}(x, y)wz^2 + A_{13}(x, y)z^2 + A_{21}(x, y)zw^2 \\ &+ A_{22}(x, y)wz + A_{23}(x, y)z + A_{31}(x, y)w^2 + A_{32}(x, y)w + A_{33}(x, y)] \end{aligned} \tag{1.7}$$

which is not assumed to satisfy cyclic property (1.1) and in which $A_{ij}(x, y)$'s are unknown functions to be determined. It is straightforward to check that the equation $I(w, x, y, z) =$

$I(w', x', y', z') = I(x, y, z, z')$ leads to a quadratic equation in z' which can be rewritten in a factorized form

$$\left[z' - w\frac{A_{11}(x,y)z^2 + A_{21}(x,y)z + A_{31}(x,y)}{A_{11}(y,z)x^2 + A_{12}(y,z)x + A_{13}(y,z)}\right] \times \left[-1 + \frac{1}{wz'}\frac{A_{13}(x,y)z^2 + A_{23}(x,y)z + A_{33}(x,y)}{A_{11}(y,z)x^2 + A_{12}(y,z)x + A_{13}(y,z)}\right] = 0 \tag{1.8}$$

provided the following conditions

$$A_{12}(x,y)z^2 - A_{21}(y,z)x^2 + A_{32}(x,y) - A_{23}(y,z) = A_{22}(y,z)x - A_{22}(x,y)z, \tag{1.9}$$

$$[A_{11}(x,y)z^2 + A_{21}(x,y)z + A_{31}(x,y)][A_{13}(x,y)z^2 + A_{23}(x,y)z + A_{33}(x,y)] = [A_{11}(y,z)x^2 + A_{12}(y,z)x + A_{13}(y,z)][A_{31}(y,z)x^2 + A_{32}(y,z)x + A_{33}(y,z)] \tag{1.10}$$

hold. Thus it is clear from equation (1.8) that under the conditions given in (1.9) and (1.10), $I(w, x, y, z)$ is an integral of an F-type mapping

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = wF(x, y, z),$$

with

$$F(x,y,z) = \frac{A_{11}(x,y)z^2 + A_{21}(x,y)z + A_{31}(x,y)}{A_{11}(y,z)x^2 + A_{12}(y,z)x + A_{13}(y,z)} \tag{1.11a}$$

$$= \frac{A_{31}(y,z)x^2 + A_{32}(y,z)x + A_{33}(y,z)}{A_{13}(x,y)z^2 + A_{23}(x,y)z + A_{33}(x,y)}. \tag{1.11b}$$

Furthermore $I(w, x, y, z)$ is also an integral of a G-type mapping

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = \frac{1}{w}G(x, y, z)$$

with

$$G(x,y,z) = \frac{A_{13}(x,y)z^2 + A_{23}(x,y)z + A_{33}(x,y)}{A_{11}(y,z)x^2 + A_{12}(y,z)x + A_{13}(y,z)} \tag{1.12a}$$

$$= \frac{A_{31}(y,z)x^2 + A_{32}(y,z)x + A_{33}(y,z)}{A_{11}(x,y)z^2 + A_{21}(x,y)z + A_{31}(x,y)}. \tag{1.12b}$$

We wish to mention that, in line with equation (1.6), the G-type of mapping will be referred to as *dual mapping* of the F-type of mapping (1.11) because of the factorization (1.8).

The plan of this paper is as follows. In section 2, we consider F-type of mapping (1.11) and investigate which conditions on the functions $A_{ij}(x, y)$ should be imposed to have three independent integrals. In section 3, we present seven families of mappings admitting three independent integrals. In section 4, we identify six families of four-dimensional mappings possessing two independent integrals. In section 5, we construct dual mappings of type G associated with the super integrable mappings derived in section 3. In section 6, we give a brief summary of our investigations, a discussion of the possible reductions to three-dimensional mappings and a comparison to mappings in the recent literature and a final comment concerning possible extensions and applications.

2. Conditions for F-type mappings to have three integrals

Consider an F-type mapping having the form

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = w\left(\frac{z}{x}\right)^{\nu} \frac{[f_1(y)xz + f_2(y)x + f_3(y)z + f_4(y)]}{[f_5(y)xz + f_6(y)x + f_7(y)z + f_8(y)]}, \tag{2.1}$$

where $f_i(y)$'s are unknown functions to be determined. Then from equation (1.11a) we find

$$A_{11}(x, y) = [xf_1(y) + f_3(y)]h_1(x, y), \tag{2.2a}$$

$$A_{31}(x, y) = [xf_2(y) + f_4(y)]h_2(x, y), \tag{2.2b}$$

$$A_{11}(y, z) = [zf_5(y) + f_6(y)]h_3(y, z), \tag{2.2c}$$

$$A_{13}(y, z) = [zf_7(y) + f_8(y)]h_4(y, z), \tag{2.2d}$$

$$A_{21}(x, y) = [xf_2(y) + f_4(y)]h_1(x, y) + [xf_1(y) + f_3(y)]h_2(x, y), \tag{2.2e}$$

$$A_{12}(y, z) = [zf_7(y) + f_8(y)]h_3(y, z) + [zf_5(y) + f_6(y)]h_4(y, z), \tag{2.2f}$$

where

$$\left(\frac{z}{x}\right)^{\nu} = \frac{h_1(x, y)z + h_2(x, y)}{h_3(y, z)x + h_4(y, z)}. \tag{2.3}$$

Similarly from equation (1.11b) we find

$$A_{31}(y, z) = [zf_1(y) + f_2(y)]h_5(y, z), \tag{2.4a}$$

$$A_{33}(y, z) = [zf_3(y) + f_4(y)]h_6(y, z), \tag{2.4b}$$

$$A_{13}(x, y) = [xf_5(y) + f_7(y))]h_7(x, y), \tag{2.4c}$$

$$A_{33}(x, y) = [xf_6(y) + f_8(y)]h_8(x, y), \tag{2.4d}$$

$$A_{32}(y, z) = [zf_3(y) + f_4(y)]h_5(y, z) + [zf_1(y) + f_2(y)]h_6(y, z), \tag{2.4e}$$

$$A_{23}(x, y) = [xf_6(y) + f_8(y)]h_7(x, y) + [xf_5(y) + f_7(y)]h_8(x, y), \tag{2.4f}$$

where

$$\left(\frac{z}{x}\right)^{\nu} = \frac{h_5(y, z)x + h_6(y, z)}{h_7(x, y)z + h_8(x, y)}. \tag{2.5}$$

Noting that

$$\frac{\partial^2}{\partial x \partial z}[z^{\nu}h_4(y, z) - x^{\nu}h_2(x, y)] = 0$$

the solution of equation (2.3) is

$$\begin{aligned} h_1(x, y) &= x^{-\nu}[xh_1(y) + l_1(y)], & h_2(x, y) &= x^{-\nu}[xl_3(y) + h_2(y)], \\ h_3(x, y) &= y^{-\nu}[yh_1(x) + l_3(x)], & h_4(x, y) &= y^{-\nu}[yl_1(x) + h_2(x)]. \end{aligned} \tag{2.6}$$

Similarly the solution of equation (2.5) can be written as

$$\begin{aligned} h_5(x, y) &= y^{\nu}[yh_5(x) + l_5(x)], & h_6(x, y) &= y^{\nu}[yl_7(x) + h_6(x)], \\ h_7(x, y) &= x^{\nu}[xh_5(y) + l_7(y)], & h_8(x, y) &= x^{\nu}[xl_5(y) + h_6(y)]. \end{aligned} \tag{2.7}$$

From equations (2.2) and (2.4) we have

$$A_{11}(y,z) = [yf_1(z) + f_3(z)]h_1(y,z) = [zf_5(y) + f_6(y)]h_3(y,z), \tag{2.8a}$$

$$A_{33}(y,z) = [zf_3(y) + f_4(y)]h_6(y,z) = [yf_6(z) + f_8(z)]h_8(y,z), \tag{2.8b}$$

$$A_{31}(y,z) = [yf_2(z) + f_4(z)]h_2(y,z) = [zf_1(y) + f_2(y)]h_5(y,z), \tag{2.8c}$$

$$A_{13}(y,z) = [yf_5(z) + f_7(z)]h_7(y,z) = [zf_7(y) + f_8(y)]h_4(y,z). \tag{2.8d}$$

Taking for simplicity $l_1(y) = l_3(y) = l_5(y) = l_7(y) = 0$ it is clear that if we have solutions of equations (2.6)–(2.8) satisfying the conditions (1.9) as well then the mapping (2.1) has three different integrals corresponding to the terms with $h_1(y)$, $h_6(y)$ and $\{h_2(y), h_5(y)\}$ respectively in the expression for $I(w,x,y,z)$ given in equation (1.7).

To work out the condition (1.9) we first note that the mixed second derivative with respect to x and z of the left-hand side of (1.9) divided by xyz must be zero. It follows that the expressions $\frac{A_{12}(x,y)}{xy}$ and $\frac{A_{32}(x,y)}{xy}$ must be linear combinations of the terms x^δ, $\delta = -1, 0, 1$, that is

$$\frac{A_{12}(x,y)}{xy} = \frac{a_1(y)x^2 + a_2(y)x + a_3(y)}{x}, \qquad \frac{A_{32}(x,y)}{xy} = \frac{b_1(y)x^2 + b_2(y)x + b_3(y)}{x} \tag{2.9a}$$

$$\frac{A_{21}(y,z)}{yz} = \frac{c_1(y)z^2 + c_2(y)z + c_3(y)}{z}, \qquad \frac{A_{23}(y,z)}{yz} = \frac{d_1(y)z^2 + d_2(y)z + d_3(y)}{z}, \tag{2.9b}$$

where $a_i(y), b_i(y), c_i(z)$ and $d_i(z)$, $i = 1,2,3$ are unknowns functions to be determined. Substituting equations (2.9a) and (2.9b) in equation (1.9) we find that the consistency requires the following conditions

$$c_1(y) = a_1(y), \qquad c_3(y) = b_1(y), \qquad d_1(y) = a_3(y), \qquad d_3(y) = b_3(y) \tag{2.10}$$

and

$$a_2(y) = \frac{a_{20} + a_{21}y + a_{22}y^2}{y}, \qquad b_2(y) = \frac{b_{20} + b_{21}y + b_{22}y^2}{y},$$
$$c_2(y) = \frac{a_{20} + a_{21}y + b_{22}y^2}{y}, \qquad d_2(y) = \frac{b_{20} + b_{21}y + a_{22}y^2}{y}, \tag{2.11}$$

where $a_{20}, a_{21}, a_{22}, b_{20}, b_{21}$ and b_{22} are unknown constants. As it will turn out the conditions (2.9)–(2.11) impose rather severe restrictions to obtain super integrable mappings.

3. Seven families of F-type of mappings

We mention here that equation (2.8) can have solutions with $f_i(y) = \kappa_i$, $i = 1, 2, \ldots, 8$ where all the $h_\alpha(y)$ for $\alpha = 1, 2, 5, 6$ in equations (2.6), (2.7) contain a factor $(y + a_\alpha)$ (case 3.1). Also, we have six solutions in which not all $f_i(y)$ are constants and where

$$h_1(x,y) = h_3(x,y), \qquad h_8(x,y) = h_6(x,y)x^{\nu_6}y^{-\nu_6} \tag{3.1a}$$

$$h_5(x,y) = h_7(y,x) = h_2(x,y)x^{-\nu_5}y^{\nu_2} = h_4(y,x)x^{-\nu_5}y^{\nu_2} \tag{3.1b}$$

which will be treated in cases 3.2–3.7

Case 3.1. In this case by taking $l_1(y) = l_3(y) = l_5(y) = l_7(y) = 0$ we have

$$\begin{aligned} h_1(x, y) &= h_3(y, x) = x^{1-\nu} y^{1-\nu}(y + a_1)h_1, \\ h_6(x, y) &= h_8(y, x) = x^{\nu} y^{\nu}(x + a_6)h_6 \end{aligned} \tag{3.2a}$$

$$\begin{aligned} h_2(x, y) &= h_4(y, x) = x^{-\nu} y^{1+\nu}(y + a_2)h_2, \\ h_5(x, y) &= h_7(y, x) = x^{-\nu} y^{1+\nu}(x + a_5)h_5. \end{aligned} \tag{3.2b}$$

From equation (2.8) we find that

$$\begin{aligned} f_1(y) &= f_5(y) = \kappa_1, \qquad f_2(y) = f_7(y) = \kappa_2, \\ f_3(y) &= f_6(y) = \kappa_3, \qquad f_4(y) = f_8(y) = \kappa_4, \end{aligned} \tag{3.3}$$

$$\begin{aligned} a_1 &= \frac{\kappa_3}{\kappa_1}, \qquad a_2 = \frac{\kappa_2}{\kappa_1}, \qquad a_5 = \frac{\kappa_4}{\kappa_2}, \\ a_6 &= \frac{\kappa_4}{\kappa_3}, \qquad h_5 = a_2 h_2. \end{aligned} \tag{3.4}$$

In order to satisfy equations (2.9)–(2.11) we must have $\nu = 0$ and then the explicit forms for $A_{ij}(x, y)$ are as given in equation (A.1) of appendix A. Then the mapping is

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = w\frac{[\kappa_1 xz + \kappa_2 x + \kappa_3 z + \kappa_4]}{[\kappa_1 xz + \kappa_2 z + \kappa_3 x + \kappa_4]}, \tag{3.5}$$

and has three independent integrals I_1, I_2 and I_3, in which $I_1\kappa_1^{-1}$, $I_2\kappa_1^{-1}$ and $I_3\kappa_3^{-1}$ are the coefficients of the terms with h_1, h_2 and h_6 respectively in the expression for $I(w, x, y, z)$. The explicit forms of integrals are

$$\begin{aligned} I_1 = {} & wz(\kappa_1 x + \kappa_3)(\kappa_1 y + \kappa_3) + z(\kappa_1 x + \kappa_3)(\kappa_2 y + \kappa_4) + w(\kappa_1 y + \kappa_3)(\kappa_2 x + \kappa_4) \\ & + \kappa_3(\kappa_2 xy + \kappa_4(x + y)), \end{aligned} \tag{3.6a}$$

$$\begin{aligned} I_2 = {} & z(\kappa_1 x + \kappa_2)(\kappa_1 + \kappa_3 y^{-1}) + z^{-1}(\kappa_4 x^{-1} + \kappa_2)(\kappa_4 + \kappa_3 y) + w(\kappa_1 y + \kappa_2)(\kappa_1 + \kappa_3 x^{-1}) \\ & + w^{-1}(\kappa_4 y^{-1} + \kappa_2)[(\kappa_4 + \kappa_3 x) + z(\kappa_1 x + \kappa_2)] + z^{-1}w(\kappa_4 x^{-1} + \kappa_2)(\kappa_2 + \kappa_1 y) \\ & + \kappa_2[\kappa_1(x + y) + \kappa_4(x^{-1} + y^{-1}) + \kappa_3(yx^{-1} + xy^{-1})], \end{aligned} \tag{3.6b}$$

$$\begin{aligned} I_3 = {} & w^{-1}z^{-1}(\kappa_4 x^{-1} + \kappa_3)(\kappa_4 y^{-1} + \kappa_3) + z^{-1}(\kappa_4 x^{-1} + \kappa_3)(\kappa_2 y^{-1} + \kappa_1) \\ & + w^{-1}(\kappa_4 y^{-1} + \kappa_3)(\kappa_2 x^{-1} + \kappa_1) + \kappa_3(\kappa_2 x^{-1}y^{-1} + \kappa_1(x^{-1} + y^{-1})). \end{aligned} \tag{3.6c}$$

Here it is worthwhile to note the mapping (3.5) is invariant under the transformation S consisting of taking the inverse of w, x, y, z and the interchange of κ_1 and κ_4, that is,

$$S(w, x, y, z, \kappa_1, \kappa_2, \kappa_3, \kappa_4) = (w^{-1}, x^{-1}, y^{-1}, z^{-1}, \kappa_4, \kappa_2, \kappa_3, \kappa_1). \tag{3.7}$$

This means that if $I(w, x, y, z, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ is an integral, then also

$$SI(w, x, y, z, \kappa_1, \kappa_2, \kappa_3, \kappa_4) = I(w^{-1}, x^{-1}, y^{-1}, z^{-1}, \kappa_4, \kappa_2, \kappa_3, \kappa_1) \tag{3.8}$$

is an integral. From equations (3.6*a*)–(3.6*c*) we see that I_2 is a symmetric integral satisfying $S \circ I_2 = I_2$ and the asymmetric integrals I_1 and I_3 transform into each other, that is,

$$S \circ I_1 = I_3, \qquad S \circ I_3 = I_1.$$

Furthermore if $F(x, y, z) \circ F(z, y, x) = 1$ the mapping L given by (2.1) has a reversing symmetry, see [10]

$$U : w \to z, \qquad x \to y, \qquad y \to x, \qquad z \to w$$

satisfying $L \circ U \circ L = U$. It is easy to see that in (3.5), $f_1(y) = f_5(y)$, $f_4(y) = f_8(y)$, $f_2(y) = f_7(y)$, $f_3(y) = f_6(y)$ and therefore (cf equation (2.1)) $F(x, y, z) \circ F(z, y, x) = 1$. Hence the mapping (3.5) has reversing symmetry U and all the integrals I_1, I_2 and I_3 in this case are invariant under U.

Case 3.2. In this case $\nu_2 = \nu_5 = \nu_6 = 0$ and equations (3.1) with (2.6) and (2.7) have the solution

$$\begin{aligned} h_1(x, y) &= h_3(x, y) = (h_1 xy + m_1(x + y) + n_1)x^{-\nu} y^{-\nu} \\ h_6(x, y) &= h_8(x, y) = (h_6 + n_2(x + y) + n_1 xy)x^{\nu} y^{\nu} \\ h_2(x, y) &= h_5(x, y) = h_7(y, x) = h_4(y, x) = (n_1 x + h_2 y + m_1 xy + n_2)x^{-\nu} y^{\nu}. \end{aligned} \tag{3.9}$$

From equation (2.8) we have the solution

$$f_1(y) = f_5(y) = \kappa_1 y + \lambda_1, \qquad f_4(y) = f_8(y) = \lambda_1 y + \kappa_1, \tag{3.10a}$$

$$f_2(y) = f_7(y) = \kappa_1(y + 1), \qquad f_3(y) = f_6(y) = \lambda_1(y + 1). \tag{3.10b}$$

In order to satisfy the conditions (2.9)–(2.11) we must have $\nu = 0$ and $m_1 = n_1 = n_2$. The explicit expressions for the $A_{ij}(x, y)$ in this case are given in equation (A.2) of appendix A. Then the mapping

$$\begin{aligned} &w' = x, \qquad x' = y, \qquad y' = z, \\ &z' = w\frac{(\kappa_1 y + \lambda_1)xz + (\kappa_1 x + \lambda_1 z)(y + 1) + (\lambda_1 y + \kappa_1)}{(\kappa_1 y + \lambda_1)xz + (\kappa_1 z + \lambda_1 x)(y + 1) + (\lambda_1 y + \kappa_1)}. \end{aligned} \tag{3.11}$$

has three independent integrals corresponding with h_1, h_2 and h_6 in the expression (1.7) for $I(w, x, y, z)$

$$I_1 = \kappa_1[wxyz + xy(w + z) + (yz + wx + xy) + w + x + y + z] + \lambda_1[wz(x + y + 1) + xz + wy], \tag{3.12a}$$

$$\begin{aligned} I_2 = {} & \kappa_1[xz + wy + w^{-1}y^{-1} + x^{-1}z^{-1} + w^{-1}xz + wyz^{-1} + w^{-1}y^{-1}z + wx^{-1}z^{-1} + w^{-1}z + wz^{-1}] \\ & + \lambda_1[w + x + y + z + w^{-1} + x^{-1} + y^{-1} + z^{-1} + (x + z)y^{-1} \\ & + (x^{-1} + z^{-1})y + xzw^{-1}y^{-1} + wyx^{-1}z^{-1} + wx^{-1} + xw^{-1} + xy^{-1}z \\ & + x^{-1}yz^{-1} + wx^{-1}y + w^{-1}xy^{-1}], \end{aligned} \tag{3.12b}$$

$$\begin{aligned} I_3 = {} & \kappa_1[w^{-1}x^{-1}y^{-1}z^{-1} + x^{-1}y^{-1}(w^{-1} + z^{-1}) + (y^{-1}z^{-1} + w^{-1}x^{-1} + x^{-1}y^{-1}) \\ & + (w^{-1} + x^{-1} + y^{-1} + z^{-1})] + \lambda_1[w^{-1}z^{-1}(x^{-1} + y^{-1} + 1) + x^{-1}z^{-1} + w^{-1}y^{-1}]. \end{aligned} \tag{3.12c}$$

The terms with $n_1 \neq 0$ in $I(w, x, y, z)$ of (1.7) give an another dependent integral of the mapping.

Case 3.3. Here $\nu_5 = 0$, $\nu_2 = \nu_6 = 1$. From equations (3.1), (2.6) and (2.7) we obtain the solution

$$\begin{aligned}
h_1(x, y) &= h_3(x, y) = (h_1 + n_1x^{-1} + n_1y^{-1})x^{1-\nu}y^{1-\nu}\\
h_6(x, y) &= h_8(y, x) = (h_6 + n_1x + n_1y)x^{\nu-1}y^{\nu}\\
h_2(x, y) &= h_4(y, x) = (h_2yx^{-1} + n_1)x^{1-\nu}y^{\nu-1}\\
h_5(x, y) &= h_7(y, x) = (h_2yx^{-1} + n_1)x^{1-\nu}y^{\nu}.
\end{aligned} \tag{3.13}$$

Equations (2.8) have the solution

$$\begin{aligned}
&f_1(y) = \lambda_1, && f_2(y) = \kappa_2y, && f_3(y) = \lambda_5y, && f_4(y) = \lambda_1y^2,\\
&f_5(y) = \lambda_5, && f_6(y) = \lambda_1y, && f_7(y) = \kappa_7y, && f_8(y) = \lambda_5y^2.
\end{aligned} \tag{3.14}$$

In order to satisfy equations (2.9)–(2.11) we must have

$$\lambda_5 = \lambda_1, \qquad \kappa_7 = \kappa_2, \qquad \nu = 1 \tag{3.15}$$

and so the explicit forms of $A_{ij}(x, y)$ read as given in equation (A.3) of appendix A. Then the mapping

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = w\left(\frac{z}{x}\right)\frac{[\lambda_1(xz + yz + y^2) + \kappa_2xy]}{[\kappa_2yz + \lambda_1(xy + xz + y^2)]} \tag{3.16}$$

has three independent integrals given by the terms h_1, h_2 and h_6 in expression (1.7) for $I(w, x, y, z)$ in which we take $n_1 = 0$. The explicit forms of the integrals are

$$I_1 = \kappa_2[w + x + y + z] + \lambda_1[wyx^{-1} + xzy^{-1} + wzy^{-1} + wzx^{-1}], \tag{3.17a}$$

$$\begin{aligned}
I_2 = \kappa_2[xzw^{-1}y^{-1} + wyx^{-1}z^{-1}] + \lambda_1[&xzy^{-2} + y^2x^{-1}z^{-1} + wyx^{-2} + x^2w^{-1}y^{-1}\\
&+ zy^{-1} + yz^{-1} + wx^{-1} + xw^{-1} + yx^{-1} + xy^{-1} + x^2zw^{-1}y^{-2} + wy^2x^{-2}z^{-1}],
\end{aligned} \tag{3.17b}$$

$$I_3 = \kappa_2[w^{-1} + x^{-1} + y^{-1} + z^{-1}] + \lambda_1[xw^{-1}z^{-1} + yw^{-1}z^{-1} + xw^{-1}y^{-1} + yx^{-1}z^{-1}]. \tag{3.17c}$$

The terms with $n_1 \neq 0$ in $I(w, x, y, z)$ give an another (dependent) integral.

Case 3.4. In this case, $\nu_5 = 0$, $\nu_2 = \nu_6 = -1$. From equations (3.1), (2.6) and (2.7) we obtain the solution

$$\begin{aligned}
h_1(x, y) &= h_3(x, y) = (h_1x^2y^2 + n_1xy)x^{-1-\nu}y^{-1-\nu}\\
h_6(x, y) &= h_8(y, x) = (h_6y^{-1} + L_5x)x^{1+\nu}y^{1+\nu}\\
h_2(x, y) &= h_4(y, x) = (h_2xy + n_1x^2y + L_5x)x^{-1-\nu}y^{1+\nu}\\
h_5(x, y) &= h_7(y, x) = (h_2x + n_1x^2 + L_5xy^{-1})x^{-1-\nu}y^{1+\nu}.
\end{aligned} \tag{3.18}$$

Equations (2.8) have the solution

$$\begin{aligned}
&f_1(y) = f_5(y) = \kappa_1y, && f_2(y) = f_7(y) = \kappa_1,\\
&f_3(y) = f_6(y) = \lambda_3, && f_4(y) = f_8(y) = \kappa_1y^{-1}.
\end{aligned} \tag{3.19}$$

In order to satisfy equations (2.9)–(2.11) we must have $\nu = -1$ and $L_5 = n_1$ and in that case the explicit forms of $A_{ij}(x, y)$ are given in equation (A.4) of appendix A. Then the mapping

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = w\left(\frac{x}{z}\right)\frac{[\kappa_1xy^2z + \lambda_3yz + \kappa_1xy + \kappa_1]}{[\kappa_1xy^2z + \kappa_1yz + \lambda_3xy + \kappa_1]} \tag{3.20}$$

has three independent integrals given by

$$I_1 = \kappa_1[xy^2z + wx^2y^2z + yz + wx^2y + wx + xy] + \lambda_3 wxyz, \tag{3.21a}$$

$$I_2 = \kappa_1[wxy + xyz + w^{-1}x^{-1}y^{-1} + x^{-1}y^{-1}z^{-1} + yzw^{-1} + wy^{-1}z^{-1} + zx^{-1}w^{-1} + xwz^{-1}] \\ + \lambda_3[w + x + y + z + w^{-1} + x^{-1} + y^{-1} + z^{-1}], \tag{3.21b}$$

$$I_3 = \kappa_1[w^{-1}x^{-1} + x^{-1}y^{-1} + y^{-1}z^{-1} + w^{-1}y^{-1}x^{-2} + x^{-1}y^{-2}z^{-1} + w^{-1}x^{-2}y^{-2}z^{-1}] \\ + \lambda_3 w^{-1}x^{-1}y^{-1}z^{-1}. \tag{3.21c}$$

Next we consider the case $\nu_5 = -1$, $\nu_2 = 0$, $\nu_6 = -1$. In that case equations (3.1), (2.6) and (2.7) have the solution

$$\begin{aligned} h_1(x, y) &= h_3(x, y) = (h_1xy + m_1(x + y) + n_1)x^{-\nu}y^{-\nu} \\ h_6(x, y) &= h_8(y, x) = (h_6 + n_2(x + y) + n_1xy)x^{1+\nu}y^{\nu} \\ h_2(x, y) &= h_4(y, x) = (h_2y + m_1xy + n_1x + n_2)x^{-\nu}y^{\nu} \\ h_5(x, y) &= h_7(y, x) = (n_1x + h_2y + m_1xy + n_2)x^{1-\nu}y^{\nu}. \end{aligned} \tag{3.22}$$

Equations (2.8) have the solution

$$\begin{aligned} &f_1(y) = f_4(y) = f_5(y) = f_8(y) = 0, \qquad f_2(y) = \lambda_2, \\ &f_7(y) = \lambda_7, \qquad f_3(y) = f_6(y) = \lambda_3. \end{aligned} \tag{3.23}$$

The consistency conditions of equations (2.9)–(2.11) hold for the following two possibilities

$$\lambda_7 = \lambda_2, \qquad m_1 = n_2 = 0, \qquad \nu = 0 \tag{3.24}$$

and

$$\lambda_7 = \lambda_2, \qquad m_1 = n_1 = n_2 = 0, \qquad \nu = -1 \tag{3.25}$$

which will be discussed below separately as cases 3.5 and 3.6.

Case 3.5. For equations (3.24) we have the following explicit expressions for $A_{ij}(x, y)$ which are given by equation (A.5) of appendix A. Then the mapping

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = w\frac{[\lambda_3 z + \lambda_2 x]}{[\lambda_3 x + \lambda_2 z]} \tag{3.26}$$

has the following three integrals

$$I_1 = \lambda_3 wz + \lambda_2[yz + wx + xy], \tag{3.27a}$$

$$I_2 = \lambda_3[wx^{-1} + xw^{-1} + xy^{-1} + yx^{-1} + yz^{-1} + zy^{-1}] + \lambda_2[wz^{-1} + zw^{-1}], \tag{3.27b}$$

$$I_3 = \lambda_2[w^{-1}x^{-1} + x^{-1}y^{-1} + y^{-1}z^{-1}] + \lambda_3 w^{-1}z^{-1}. \tag{3.27c}$$

Case 3.6. In the case of equation (3.25), the explicit expressions for $A_{ij}(x, y)$ are given in equation (A.6) of appendix A. Then the mapping

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = w\left(\frac{x}{z}\right)\frac{[\lambda_3 z + \lambda_2 x]}{[\lambda_3 x + \lambda_2 z]} \tag{3.28}$$

has three independent integrals given by

$$I_1 = \lambda_3 wxyz + \lambda_2[xy^2z + wx^2y], \tag{3.29a}$$

$$I_2 = \lambda_2[wxy^{-1}z^{-1} + yzw^{-1}x^{-1}] + \lambda_3[wy^{-1} + yw^{-1} + xz^{-1} + zx^{-1}], \tag{3.29b}$$

$$I_3 = \lambda_2[w^{-1}x^{-2}y^{-1} + x^{-1}y^{-2}z^{-1}] + \lambda_3 w^{-1}x^{-1}y^{-1}z^{-1}. \tag{3.29c}$$

Case 3.7. In this case, $\nu_5 = 1$, $\nu_2 = -2$, $\nu_6 = -1$. Then equations (3.1), (2.6) and (2.7) have a solution with $l_1(y) = l_3(y) = l_5(y) = l_7(y) = 0$ and

$$\begin{aligned} h_1(x, y) &= h_3(x, y) = x^{1-\nu}y^{1-\nu}h_1, & h_6(x, y) &= h_8(y, x) = x^{1+\nu}y^{\nu}h_6 \\ h_2(x, y) &= h_4(y, x) = x^{-\nu}y^{\nu+3}h_2, & h_5(x, y) &= h_7(y, x) = x^{-1-\nu}y^{1+\nu}h_2. \end{aligned} \tag{3.30}$$

Equations (2.8) have the solution

$$\begin{aligned} f_1(y) &= f_5(y) = \kappa_1 y, & f_3(y) &= f_6(y) = \lambda_3, \\ f_4(y) &= f_8(y) = \kappa_1 y^{-1}, & f_2(y) &= f_7(y) = 0. \end{aligned} \tag{3.31}$$

Furthermore the consistency of equations (2.9)–(2.11) requires that $\nu = -2$. As a result we have the following explicit forms of $A_{ij}(x, y)$ as given by (A.7) of appendix A. Then the mapping

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = w\left(\frac{x}{z}\right)^2 \frac{[\kappa_1 x y^2 z + \lambda_3 yz + \kappa_1]}{[\kappa_1 x y^2 z + \lambda_3 xy + \kappa_1]} \tag{3.32}$$

has three independent integrals

$$I_1 = \lambda_3 w x^2 y^2 z + \kappa_1 xy(yz + wx + wx^2y^2z), \tag{3.33a}$$

$$\begin{aligned} I_2 = \kappa_1[&xy^2z + wx^2y + x^{-1}y^{-2}z^{-1} + w^{-1}x^{-2}y^{-1} + wxy^{-1}z^{-1} + yzw^{-1}x^{-1}] \\ &+ \lambda_3[wx + yz + xy + w^{-1}x^{-1} + y^{-1}z^{-1} + x^{-1}y^{-1}], \end{aligned} \tag{3.33b}$$

$$I_3 = \kappa_1[w^{-1}x^{-3}y^{-3}z^{-1} + x^{-1}y^{-2}z^{-1} + w^{-1}x^{-2}y^{-1}] + \lambda_3 w^{-1}x^{-2}y^{-2}z^{-1}. \tag{3.33c}$$

Considering the symmetries of the mappings derived in cases 3.2–3.7 we see that in all cases the mappings have the form (2.1) with

$$f_1(y) = f_5(y), \qquad f_4(y) = f_8(y), \qquad f_2(y) = f_7(y), \qquad f_3(y) = f_6(y), \tag{3.34}$$

and

$$\begin{aligned} f_1(y) &= f_4(y^{-1})y^{\lambda}, & f_2(y) &= f_2(y^{-1})y^{\lambda}, \\ f_3(y) &= f_3(y^{-1})y^{\lambda}, & f_4(y) &= f_1(y^{-1})y^{\lambda}. \end{aligned} \tag{3.35}$$

Equation (3.34) implies that all mappings have reversing symmetry U,

$$U : w \to z, \qquad x \to y, \qquad y \to x, \qquad z \to w$$

satisfying $L \circ U \circ L = U$, cf [10], and equation (3.35) implies that all mappings are invariant under taking the inverse I, that is

$$I(w, x, y, z) = I(w^{-1}, x^{-1}, y^{-1}, z^{-1})$$

In all these cases, the integrals I_1, I_2 and I_3 are invariant under the reversing symmetry U, the integral I_2 is symmetric under the inversion $I \circ I_2 = I_2$ and the integrals I_1 and I_3 are asymmetric and are transformed into each other, that is

$$I \circ I_1 = I_3, \qquad I \circ I_3 = I_1.$$

Finally it is worthwhile to note that in cases 3.3–3.7 the mappings can be reduced to lower dimensional mappings which will be discussed in section 6.

4. *F*-type mappings to have two integrals

In this section, we will restrict ourselves to solutions of equations (2.6), (2.7) and (3.1) with $l_1(y) = l_3(y) = l_5(y) = l_7(y) = 0$ and

$$h_1(x, y) = h_2(x, y) = x^{1-\nu} y^{1-\nu} h_1, \qquad h_6(x, y) = h_8(y, x) = x^{\nu-\nu_6} y^{\nu} h_6$$
$$h_2(x, y) = h_4(y, x) = x^{\nu} y^{1+\nu-\nu_2} h_2, \qquad h_5(x, y) = h_7(y, x) = x^{-\nu-\nu_5} y^{1+\nu} h_2.$$

Assuming that a linear relation exists among h_1, h_2 and h_6 we can derive F-type of mappings with two integrals in the following six cases:

Case 4.1. $h_2 = 0, \nu_6 = 0, \nu = 0.$

Case 4.2. $h_2 = 0, \nu_6 = -1, \nu = -1.$

Case 4.3. $h_2 = 0, \nu_6 = -1, \nu = -2.$

Case 4.4. $h_1 = h_6, \nu_5 = -1, \nu_2 = 0, \nu_6 = -1, \nu = 1.$

Case 4.5. $h_1 = h_6, \nu_5 = 1, \nu_2 = 2, \nu_6 = 3, \nu = 2.$

Case 4.6. $h_6 = 0, \nu_5 = 1, \nu_2 = -1, \nu = -1.$

Case 4.1. In this case $h_2 = 0, \nu_6 = 0$. From equations (2.8) we obtain

$$f_1(y) = f_5(y) = \kappa_1 y + \lambda_1, \qquad f_3(y) = f_6(y) = \lambda_1 y + \lambda_3, \qquad f_4(y) = f_8(y) = \lambda_3 y + \lambda_4. \tag{4.1}$$

Making use of the above in equations (2.9)–(2.11) with $h_2 = 0$ we find

$$\nu = 0, \qquad f_2(y) = f_7(y) = \kappa_2 y + \lambda_2. \tag{4.2}$$

As a consequence the functions $A_{ij}(x, y)$ take the following forms given by equation (A.8) of appendix A. Then the mapping becomes

$$\begin{aligned} &w' = x, \qquad x' = y, \qquad y' = z, \\ &z' = w \frac{[\kappa_2 xy + \lambda_2 x + \lambda_3 y + \lambda_4] + z[\kappa_1 xy + \lambda_1 (x + y) + \lambda_3]}{[\kappa_2 yz + \lambda_3 y + \lambda_2 z + \lambda_4] + x[\kappa_1 yz + \lambda_1 (y + z) + \lambda_3]} \end{aligned} \tag{4.3}$$

which admits two independent integrals

$$\begin{aligned} I_1 = {} & \lambda_4(w + x + y + z) + \lambda_2(wx + xy + yz) + \lambda_3(wy + wz + xz) \\ & + \lambda_1(wxz + wyz) + \kappa_2(wxy + xyz) + \kappa_1 wxyz, \end{aligned} \tag{4.4a}$$

$$\begin{aligned} I_3 = {} & \kappa_1[w^{-1} + x^{-1} + y^{-1} + z^{-1}] + \kappa_2[w^{-1}x^{-1} + x^{-1}y^{-1} + y^{-1}z^{-1}] \\ & + \lambda_1[w^{-1}y^{-1} + w^{-1}z^{-1} + x^{-1}z^{-1}] + \lambda_2[w^{-1}x^{-1}y^{-1} + x^{-1}y^{-1}z^{-1}] \\ & + \lambda_3[w^{-1}x^{-1}z^{-1} + w^{-1}y^{-1}z^{-1}] + \lambda_4 w^{-1}x^{-1}y^{-1}z^{-1}. \end{aligned} \tag{4.4b}$$

Next we consider the case $h_2 = 0, \nu_6 = -1$. In that case that $h_2 = 0$, equations (2.8) lead to

$$f_1(y) = f_5(y) = \kappa_1 y, \qquad f_3(y) = f_6(y) = \lambda_3, \qquad f_4(y) = f_8(y) = \lambda_4 y^{-1}. \tag{4.5}$$

Then equations (2.9)–(2.11) have two distinct solutions. They are

$$\nu = -1, \qquad f_2(y) = f_7(y) = \lambda_2 \quad \text{(case 4.2)} \tag{4.6}$$

and

$$\nu = -2, \qquad f_2(y) = f_7(y) = 0 \quad \text{(case 4.3)} \tag{4.7}$$

which will be discussed below separately as cases 4.2 and 4.3.

Case 4.2. From equation (4.6), we have the following explicit forms of $A_{ij}(x, y)$ given by equation (A.9) of appendix A. Then the mapping

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = \frac{wx}{z} \frac{[\kappa_1 xy^2z + \lambda_3 yz + \lambda_2 xy + \lambda_4]}{[\kappa_1 xy^2z + \lambda_3 xy + \lambda_2 yz + \lambda_4]} \tag{4.8}$$

admits the following two integrals

$$I_1 = \kappa_1 wx^2y^2z + \lambda_3 wxyz + \lambda_2[xy^2z + wx^2y] + \lambda_4[yz + wx + xy], \tag{4.9a}$$

$$I_3 = \kappa_1[w^{-1}x^{-1} + y^{-1}z^{-1} + x^{-1}y^{-1}] + \lambda_2[x^{-1}y^{-2}z^{-1} + w^{-1}x^{-2}y^{-1}] + \lambda_3 w^{-1}x^{-1}y^{-1}z^{-1} + \lambda_4 w^{-1}x^{-2}y^{-2}z^{-1}. \tag{4.9b}$$

Case 4.3. For equation (4.7) the explicit forms for $A_{ij}(x, y)$ are given in equations (A.10) of appendix A. Then the mapping

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = w\left(\frac{x}{z}\right)^2 \frac{[\kappa_1 xy^2z + \lambda_3 yz + \lambda_4]}{[\kappa_1 xy^2z + \lambda_3 xy + \lambda_4]} \tag{4.10}$$

possesses two independent integrals

$$I_1 = \kappa_1 wx^3y^3z + \lambda_3 wx^2y^2z + \lambda_4[xy^2z + wx^2y], \tag{4.11a}$$

$$I_3 = \kappa_1[w^{-1}x^{-2}y^{-1} + x^{-1}y^{-2}z^{-1}] + \lambda_3 w^{-1}x^{-2}y^{-2}z^{-1} + \lambda_4 w^{-1}x^{-3}y^{-3}z^{-1}. \tag{4.11b}$$

Case 4.4. Here $\nu_5 = -1, \nu_2 = 0, \nu_6 = -1, h_2 \neq 0$. In that case equations (2.8) have the solutions

$$f_1(y) = f_4(y) = f_5(y) = f_8(y) = 0, \qquad f_3(y) = f_6(y) = \lambda_3,$$
$$f_2(y) = \lambda_2,\ f_7(y) = \lambda_7. \tag{4.12}$$

Equations (2.9)–(2.11) have solutions with $\lambda_2 = \lambda_7$ and general h_1, h_2, h_6 for $\nu = 0$ and $\nu = -1$ which have been considered in case 3.5 and case 3.6, respectively. But there is another solution with general λ_2, λ_7 and $h_1 = h_6$ for $\nu = 1$ with explicit forms of $A_{ij}(x, y)$ given by equations (A.11) of appendix A. Then the mapping

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = w\left(\frac{z}{x}\right) \frac{[\lambda_2 x + \lambda_3 z]}{[\lambda_3 x + \lambda_7 z]} \tag{4.13}$$

admits two independent integrals

$$I_1 + I_3 = \lambda_3[wzx^{-1}y^{-1} + xyw^{-1}z^{-1}] + \lambda_7[zx^{-1} + yw^{-1}] + \lambda_2[wy^{-1} + xz^{-1}], \tag{4.14a}$$

$$I_2 = \lambda_3[xzy^{-2} + y^2x^{-1}z^{-1} + wyx^{-2} + x^2w^{-1}y^{-1}] + \lambda_7 xzw^{-1}y^{-1} + \lambda_2 wyx^{-1}z^{-1}. \tag{4.14b}$$

Case 4.5. Here $h_1 = h_6, \nu_5 = 1, \nu_2 = 2, \nu_6 = 3$. In this case equations (2.8) have solutions

$$f_1(y) = f_5(y) = \kappa_1 y, \qquad f_2(y) = \kappa_2 y^2, \qquad f_7(y) = \kappa_7 y^2,$$
$$f_4(y) = f_8(y) = \kappa_1 y^3, \qquad f_3(y) = f_6(y) = 0. \tag{4.15}$$

From equations (2.9)–(2.11) we have a solution with general κ_2, κ_7 and $h_1 = h_6$ for $\nu = 2$ and the explicit forms of $A_{ij}(x, y)$ are given in equations (A.12) of appendix A. Then the mapping

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = w\left(\frac{z}{x}\right)^2 \frac{[\kappa_1 xz + \kappa_2 xy + \kappa_1 y^2]}{[\kappa_1 xz + \kappa_7 yz + \kappa_1 y^2]} \tag{4.16}$$

has two independent integrals

$$I_1 + I_3 = \kappa_1[wzx^{-1}y^{-1} + xyw^{-1}z^{-1} + xzy^{-2} + y^2x^{-1}z^{-1} + wyx^{-2} + x^2w^{-1}y^{-1}] + \kappa_7[zy^{-1} + xw^{-1} + yx^{-1}] + \kappa_2[xy^{-1} + yz^{-1} + wx^{-1}], \quad (4.17a)$$

$$I_2 = \kappa_1[xzy^{-2} + y^2x^{-1}z^{-1} + wyx^{-2} + x^2w^{-1}y^{-1} + x^3zw^{-1}y^{-3} + wy^3x^{-3}z^{-1}] + \kappa_7x^2zw^{-1}y^{-2} + \kappa_2wy^2x^{-2}z^{-1}. \quad (4.17b)$$

Case 4.6. Here $h_6 = 0$, $\nu_5 = 1$, $\nu_2 = -1$. In this case equations (2.8) have solutions

$$f_1(y) = f_5(y) = \kappa_1 y, \qquad f_4(y) = f_8(y) = \kappa_1, \qquad f_3(y) = f_6(y) = \lambda_3, \qquad f_2(y) = f_7(y) = 0. \quad (4.18)$$

Considering the conditions (2.9)–(2.11) we must have $\nu = -1$ and the explicit expressions for $A_{ij}(x, y)$'s are given in equations (A.13) of appendix A. Then the mapping

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = w\left(\frac{x}{z}\right)\frac{[\kappa_1xyz + \lambda_3z + \kappa_1]}{[\kappa_1xyz + \lambda_3x + \kappa_1]} \quad (4.19)$$

has two independent integrals of motion

$$I_1 = \kappa_1[wx^2y^2z + wxy + xyz] + \lambda_3wxyz, \quad (4.20a)$$

$$I_2 = \lambda_3[w + x + y + z + w^{-1}y^{-1} + x^{-1}z^{-1}] + \kappa_1[wxy + xyz + w^{-1}x^{-1}y^{-1} + x^{-1}y^{-1}z^{-1} + zw^{-1} + wz^{-1}]. \quad (4.20b)$$

It is appropriate to mention here that the mappings identified in cases 4.4 and 4.5 do not have the reversing symmetry U, because $\lambda_2 \neq \lambda_7$ and $\kappa_2 \neq \kappa_7$ in contrast to the mappings in cases 4.1–4.3 and 4.6. Similarly in cases 4.1–4.5 the mapping is invariant under a transformation consisting of taking the inverse of w, x, y, z combined with the interchange of two parameters occurring in the mapping, but in case 4.6 this symmetry is not there. Lower dimensional reductions of the mappings derived in this section will be discussed in section 6.

5. Seven families of dual mappings of type *G* with two integrals

Starting from the seven families of super integrable F-type of mappings treated in section 3, satisfying the conditions (1.9) and (1.10) and the factorization property (1.8) we have derived seven families of dual mappings of type G as given in (1.12). The results will be specified in cases 5.1–5.7 which are associated with cases 3.1–3.7 of section 3.

Case 5.1. From equations (B.1) of appendix B and equation (1.12), we obtain

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = \frac{1}{w}\frac{[xz(y + a_5)a_2h_2 + (y + a_6)h_6]}{[xyz(y + a_1)h_1 + y(y + a_2)h_2]}. \quad (5.1)$$

For general a_1, a_2, a_5 and a_6 with $a_2a_5 = a_1a_6$ this mapping has only one integral $I(w, x, y, z)$. However, for the following two possibilities

$$h_2 = 0, \qquad h_1 \neq 0, \qquad h_6 \neq 0, \qquad \text{(case 5.1.1)} \quad (5.2)$$

and

$$h_1 = h_6 = 0, \qquad h_2 \neq 0 \quad \text{(case 5.1.2)} \quad (5.3)$$

equation (5.1) reduces into a mapping with two independent integrals.

Case 5.1.1. Taking $h_2 = 0$ in equation (5.1) we find

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = \left(\frac{h_6}{h_1 wxyz}\right)\frac{[y + a_6]}{[y + a_1]} \tag{5.4}$$

which is an integrable and symplectic mapping identical to equation (3.23) of [7] obtained from the $z_1 = 2, z_2 = 3$ reduction of the $\Delta\Delta$mKdV difference equation. The two independent integrals are given by the terms with κ_2 and without κ_2 in the expression for $I(w, x, y, z) = I_1(w, x, y, z) + I_3(w, x, y, z)$.

Case 5.1.2. Taking $h_1 = h_6 = 0$ in equation (5.1) we find

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = \left(\frac{h_2 xz}{h_2 wy}\right)\frac{[a_2 y + a_2 a_5]}{[y + a_2]} \tag{5.5}$$

which is also an integrable and symplectic mapping equation (3.26) of [7] obtained from the $z_1 = 2, z_2 = 3$ reduction of the $\Delta\Delta$s-G difference equation. One integral is given by the terms with κ_3 in $I_2(w, x, y, z)$ and the other integral by the terms without κ_3.

Case 5.2. From equations (B.2) and (1.12), we obtain

$$\begin{aligned} &w' = x, \qquad x' = y, \qquad y' = z, \\ &z' = \frac{1}{w}\frac{[(h_2 - n_1)xz + (h_6 - n_1) + n_1(x + 1)(y + 1)(z + 1)]}{[(h_1 - n_1)xyz + (h_2 - n_1)y + n_1(x + 1)(y + 1)(z + 1)]}. \end{aligned} \tag{5.6}$$

We wish to mention here that for $n_1 \neq 0$ the mapping (5.6) has two independent integrals corresponding to the terms κ_1 and the terms with λ_1 respectively in the expression for $I(w, x, y, z)$ including the contribution arising from the terms with n_1 in the $A_{ij}(x, y)$. However, for $n_1 = 0$ the mapping (5.6) reduces to the integrable and symplectic mapping, equation (3.5) of [7] which is the $z_1 = 1, z_2 = 5$ reduction of the $\Delta\Delta$mKdV difference equation.

Case 5.3. From equations (B.3) and (1.12) we obtain

$$\begin{aligned} &w' = x, \qquad x' = y, \qquad y' = z, \\ &z' = \frac{xz}{wy}\frac{[(h_2 - n_1)xz + h_6 y + n_1(xz + yz + xy + y^2)]}{[h_1 xz + (h_2 - n_1)y + y^{-1}n_1(xz + yz + xy + y^2)]}. \end{aligned} \tag{5.7}$$

Here also for $n_1 \neq 0$ the mapping has two independent integrals corresponding to the terms κ_2 and λ_1 respectively in the expression $I(w, x, y, z)$ including the contribution from the terms with n_1 in the $A_{ij}(x, y)$. For $n_1 = 0$ the mapping (5.7) reduces to the symplectic and integrable mapping equation (3.19) of [7] which is the $z_1 = 1, z_2 = 4$ reduction of the $\Delta\Delta$s-G difference equation.

Case 5.4. From equations (B.4) and (1.12), we obtain

$$\begin{aligned} &w' = x, \qquad x' = y, \qquad y' = z, \\ &z' = \frac{1}{wxyz}\frac{[h_2 xyz + (h_6 - n_1) + n_1(xy + 1)(yz + 1)]}{[(h_1 - n_1)xyz + h_2 + n_1 y^{-1}(xy + 1)(yz + 1)]}. \end{aligned} \tag{5.8}$$

We wish to mention here that for $n_1 \neq 0$, the above mapping has two independent integrals of motion which can be obtained as the terms corresponding to κ_1 and λ_3 in $I(w, x, y, z)$ including the contributions of the terms with n_1 in $A_{ij}(x, y)$. However, for $n_1 = 0$ the mapping (5.8) reduces to the integrable and symplectic mapping (3.18) of [7] which is the $z_1 = 1, z_2 = 4$ reduction of the $\Delta\Delta$mKdV difference equation.

Case 5.5. From equations (B.5) and (1.12) we obtain

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = \frac{[h_2 xyz + n_1 y^2(x + z) + h_6 y]}{w[h_1 xyz + n_1(x + z) + h_2 y]}. \tag{5.9}$$

For $n_1 \neq 0$ this mapping, which has also been treated in some detail as motivating example in equation (32) of [9], has two independent integrals obtained by taking the terms corresponding to λ_2 and λ_3 in $I(w, x, y, z)$ including the contributions arising from the terms with n_1 in the $A_{ij}(x, y)$. However, for $n_1 = 0$ the mapping (5.9) reduces to an integrable and symplectic mapping equation (3.7) of [7] which is the $z_1 = 1, z_2 = 3$ reduction of the $\Delta\Delta$s-G difference equation.

Case 5.6. From equations (B.6) and (1.12) we obtain

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = \frac{[h_2 x y^2 z + h_6]}{wxz[h_1 x y^2 z + h_2]}, \tag{5.10}$$

which is equation (3.12) of [7].

Case 5.7. From equations (B.7) and (1.12), we obtain

$$w' = x, \qquad x' = y, \qquad y' = z, \qquad z' = \frac{[h_2 x y^2 z + h_6]}{wx^2y^2z^2[h_1 x y^2 z + h_2]}. \tag{5.11}$$

The above four-dimensional mapping has a symplectic structure characterized by the values $p = 2, \alpha = 2$ in expressions (2.19) and (2.20) of [7] for the symplectic matrix Ω. It also has two independent integrals characterized by the terms with κ_1 and λ_3 in the expressions for $I(w, x, y, z)$ given by equation (1.7).

6. Summary of results and concluding remarks

6.1. Summary of results

In section 3, we have specified seven families of super integrable mapping of type F as given by (1.5) and (1.11) which are measure preserving [10] with density $\frac{1}{wxyz}$ and in which $A_{ij}(x, y)$ are linear combinations of terms involving (independent) parameters h_1, h_2 and h_6 not occurring in the mapping. The three integrals I_1, I_2 and I_3 are obtained as the coefficients of the terms with h_1, h_2 and h_6 in the expression for $I(w, x, y, z)$. The treatment is based on a factorization property equation (1.8) in differencing the integral $I(w, x, y, z)$ which is valid under the conditions (1.9) and (1.10). (For other treatments emphasizing the role of factorization in studying integrability, see [11, 12].) In section 4, we have specified six families in cases 4.1–4.6 in which a linear relation exists between the parameters h_1, h_2 and h_6 leading to mappings of type (1.11) with two integrals. Finally in section 5, we have specified seven families of G-type of mappings as in (1.12) in cases 5.1–5.7 which are the dual mappings of the F-type of mapping in cases 3.1–3.7.

6.2. Reductions to lower dimensional mappings

From the mappings derived in sections 3 and 4, cases 3.3, 3.4 and 3.7 as well as 4.2, 4.3 and 4.5 can be reduced to three-dimensional mappings. In cases 3.5, 3.6 and 4.4, the mapping can be reduced to a two-dimensional mapping of the QRT-family. In fact, the mapping given in (case 3.3) equation (3.16) can be reduced to three-dimensional mapping in terms of the variables $W = \frac{x}{w}, X = \frac{y}{x}, Y = \frac{z}{y}$. The three-dimensional mapping has two independent integrals namely I_2 and $I_1 I_3$ obtained from equations (3.17*b*) and (3.17*a*), (3.17*c*)) in terms of (W, X, Y). It is appropriate to mention here that this three-dimensional mapping is a special case $a_3 = a_1 = b_1 = \lambda_1, b_3 = a_0 = \kappa_2$ of the (integrable) Hirota Y_1 equation [13] which has been expressed as a pair of QRT mappings [2] in [14].

Similarly the four-dimensional mapping given in (case 3.4) equation (3.20) can be reduced into a three-dimensional mapping in terms of the variables $W = wx, X = xy, Y = yz$ which is also a measure preserving one and admits two independent integrals I_1 and I_3 obtained from equations (3.21*a*) and (3.21*c*) in terms of (W, X, Y). This mapping has also been given in equations (59)–(61) of [9] with $z = 4$ which is the dual of the $z_1 = 1, z_2 = 4$ reduction of the $\Delta\Delta$mKdV equation as given in equation (3.18) of [7]. It is also the special case with $\lambda_2 = \lambda_4 = \kappa_1$ of the three-dimensional reduction with $W = wx, X = xy, Y = yz$ of equation (4.8) in case 4.2. This more general three-dimensional mapping has also been treated in [15], cf also equation (20) of [16] with $p_4 = \kappa_3, p_3 = \lambda_2, \alpha(n) = \lambda_3, \rho(n) = \lambda_4$. The three-dimensional reduction of equation (4.8) is measure preserving and has two integrals given by the expressions for I_1 and I_3 in equations (4.9*a*) and (4.9*b*) expressed in terms of (W, X, Y).

Next, the four-dimensional mapping given in (case 3.7) can be reduced to a three-dimensional mapping in terms of the variables $W = wx, X = xy, Y = yz$. This three-dimensional reduction has also been given in equation (37) of [9] with $\alpha = \lambda_3, \beta = \kappa_1$. Moreover this three-dimensional reduction is a special case of $\lambda_4 = \kappa_1$ of the three-dimensional mapping of equation (4.10). The more general three-dimensional mapping with $\lambda_4 \neq \kappa_1$ is measure preserving and has two integrals given by the integrals I_1 and I_3 in equation (4.11*a*) and (4.11*b*) in terms of (W, X, Y).

Finally, the four-dimensional mapping given in (case 4.5) equation (4.16) can be reduced to a three-dimensional mapping in terms of the variables $W = wx, X = xy, Y = yz$. This three dimensional reduction has also been given in equation (38) of [9] with $\alpha = 0, \beta = k_1, \gamma = \kappa_7, \delta = \kappa_2$. Furthermore this mapping is measure preserving and has two integrals which are given by the expressions for $I_1 + I_3$ and I_2 in equations (4.17*a*) and (4.17*b*) in terms of the variables (W, X, Y).

6.3. Super integrable and dual systems

From the results of section 5, it can be inferred that all seven families of super integrable mappings of F-type derived in section 3 can be identified as the dual mappings of known integrable systems. In fact the mapping given in equation (3.11) in case 3.1 is the dual mapping of the $z_1 = 2, z_2 = 3$ reduction of the $\Delta\Delta$mKdV difference equation and also the dual mapping of the $z_1 = 2, z_2 = 3$ reduction of the $\Delta\Delta$s-G equation. On the other hand, both reductions of the $\Delta\Delta$mKdV and $\Delta\Delta$s-G equations given in equations (3.23) and (3.26) of [7] are dual mappings of (3.11) that can be obtained by taking into account only the terms with h_1 and h_6 or only the terms with h_2 in the expression for $I(w, x, y, z)$.

The super integrable mappings derived in cases 3.2, 3.3, 3.4 and 3.5 are the dual mappings of the special cases with $n_1 = 0$ in equations (5.6), (5.7), (5.8) and (5.9). In fact from these four cases we respectively obtain

(i) the integrable and symplectic mapping (3.5) of [7] which is the $z_1 = 1, z_2 = 5$ reduction of the $\Delta\Delta$mKdV difference equation,
(ii) the integrable and symplectic mapping (3.19) of [7] which is the $z_1 = 1, z_2 = 4$ reduction of the $\Delta\Delta$s-G difference equation,
(iii) the integrable and symplectic mapping (3.18) of [7] which is the $z_1 = 1, z_2 = 4$ reduction of the $\Delta\Delta$mKdV difference equation,
(iv) the integrable and symplectic mapping (3.7) of [7] which is the $z_1 = 1, z_2 = 3$ reduction of the $\Delta\Delta$s-G difference equation.

Hence, we conclude that the super integrable mappings given in cases 3.1–3.5 can be identified as being the dual mappings of all six symplectic and integrable four-dimensional mappings that are obtained in [7] as periodic reduction of difference equations on the two-dimensional lattice as treated in [4].

On the other hand, the symplectic and integrable mappings of [7] are obtained as the duals of the super integrable mappings in cases 3.1–3.5 if we only take into account of the terms with h_1, h_2 and h_6 in the definition of $I(w, x, y, z)$. However in cases 3.2–3.5, we may also take into account another (dependent) integral characterized by the terms with n_1 in the expression for $I(w, x, y, z)$. In this way we obtain four four-dimensional mappings of type G with four parameters given in equations (5.6), (5.7), (5.8) and (5.9). Among the mappings equation (5.14) has been given before as motivating example, cf equation (32) of [9], the other mappings have been obtained by using similar constructions of dual mappings. The reported 4-parameter mappings of type G have two independent integrals given by the terms with κ_1 and λ_1 in the case of equation (5.6), κ_2 and λ_1 in the case of equation (5.7), κ_1 and λ_3 in the case of equation (5.8) and λ_2 and λ_3 in the case of equation (5.9) respectively in the expressions for $I(w, x, y, z)$ in which the extra terms due to n_1 are included. Furthermore, the mappings in cases 4.1 and 4.6, cf equations (4.3) and (4.19) provide few more examples of four-dimensional mappings (now of type F) having two integrals. Since no symplectic structure has been found for the above-mentioned six mappings, the question concerning their integrability nature remains open. One possible approach to this problem could be based on possible extensions of the investigations of [17] concerning the relation between integrability and singularity confinement criteria. However a more direct test for integrability might be developed by considering a possible four-dimensional generalization of the arithmetical approach to integrability by Roberts *et al* [18, 19].

The four-dimensional mappings given in equations (5.10) and (5.11) can be reduced to two-dimensional mappings of the QRT family [2] in terms of the variables $W = wx^2y$ and $X = xy^2z$. The two-dimensional reduction of (5.10) is the same as the two-dimensional reduction of equation (4.13) of case (4.4) in terms of the variables $W = \frac{y}{w}$, $X = \frac{z}{x}$. Also the two-dimensional reduction of equation (5.11) is the same as the two-dimensional reduction of equation (3.26) in case 3.5 in terms of the variables $W = \frac{y}{w}$, $X = \frac{z}{x}$.

6.4. LSD-type of mappings

It is interesting to note that all the mappings given in cases 3.1–3.7 contain an integral $I_1(w, x, y, z)$ which is a linear expression in terms of the variable z. From this we can apply a method of interchanging integrals and variables [20] and solve for z as a function of the value of the integral I_1 and the fields (w, x, y) and the parameters occurring in the mapping. Inserting this solution into $w' = x, x' = y, y' = z$ we obtain a three-dimensional level-set dependent (LSD) mapping that in the original four-dimensional mapping acts on the level sets of its integral I_1 as the three dimensional LSD mapping, for background, see also [21, 22] in which it was shown that each QRT map acts as a generalised McMillan mapping on the level sets of the QRT mapping, see also [9] in which a class of higher dimensional dual mKdV mappings was expressed in terms of LSD mKdV difference equations. Furthermore the three-dimensional reductions of equation (4.8) in case 4.2 and equation (4.10) in case 4.3 with $W = wx, X = xy, Y = yz$ contain an integral I_1 which is linear in y and can be reduced to two-dimensional level-set dependent mappings of the QRT family. Furthermore in cases 3.1–3.5 as well as in cases 4.4–4.6 the integral $I_2(w, x, y, z)$ can be expressed as sum of two integrals $J_1(w, x, y, z)$, $J_2(w, x, y, z)$ satisfying $J_1(x, y, z, z') = J_2(w, x, y, z)$, $J_2(x, y, z, z') = J_1(w, x, y, z)$ and containing the terms with zw^{-1}, z, w^{-1} and with wz^{-1}, z^{-1}, w respectively [23]. It would be worthwhile to investigate the problem of level-set dependent reductions of mappings containing an integral linear in one of the coordinates from a general point of view.

Using the ideas of interchanging parameters and integrals in discrete dynamical systems developed in [24] and taking the Hirota Y_1 equation [13] as a starting point, Roberts and Quispel constructed a very general 21-parameter family of integrable three-dimensional mappings [16]. It would be of interest to identify the reductions to three-dimensional mappings discussed in the present paper as special cases of this very general family.

To conclude this paper, it is worthwhile to point out that the methods developed in this paper can be extended to more general N-dimensional mappings of the type $w_{n+N} = w_n F(w_n, w_{n+1}, \ldots, w_{n+N-1})$ and $w_{n+N} = \frac{1}{w_n} G(w_n, w_{n+1}, \ldots, w_{n+N-1})$ in which the right-hand side is a rational function with numerator and denominator as in equation (2.1) being bilinear in w_{n+1} and w_{n+N-1} with coefficients depending on $(w_{n+2}, w_{n+3}, \ldots, w_{n+N-2})$. These mappings can be derived from a general integral $I(w_n, w_{n+1}, \ldots, w_{n+N-1})$ as in equation (1.7) being bilinear in $\left(w_n, w_{n+N-1}, w_n^{-1}, w_{n+N-1}^{-1}\right)$ and satisfying factorization properties as in equation (1.8). On the basis of the factorization property, these mapping have two or more integrals which can be evaluated explicitly (and possibly in special cases $(N-1)$ integrals to find new super integrable cases). The results of the above will be reported elsewhere [25]. On the other hand the factorization property is a natural expression of duality [9] leading to integrable mappings of type G, cf equation (1.3) which for even N include all periodic reductions of the integrable sine-Gordon and modified Korteweg–de Vries equations on the two-dimensional lattice [4]. Therefore it may be concluded that the methods in the present paper are independent of the dimension of the mapping and be applied quite generally to gain systematic insight in the integrability properties of higher dimensional mappings as well as interesting new cases. The higher dimensional mappings can be applied to all kinds of physical applications involving stationary, periodic and similarity solutions of evolution equations or the two-dimensional lattice (or in a good approximation more dimensional ones with interactions being predominant in one direction, see e.g. several interesting papers in [26], or difference equations occurring in exactly solvable models on two-dimensional lattice [27]).

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Appendix A.

In this appendix, we present the explicit forms for the $A_{ij}(x, y)$ in cases 3.1–3.7 and 4.1–4.6

Case 3.1. In case 3.1 with $\nu = 0$, we have

$$\begin{aligned}
&A_{11}(x, y) = xy(\kappa_1 x + \kappa_3)(y + a_1)h_1, \qquad A_{13}(x, y) = a_2 x(\kappa_1 x + \kappa_2)(y + a_5)h_2,\\
&A_{12}(x, y) = x[(\kappa_1 y + \kappa_3)(x + a_2)h_2 + y(x + a_1)(\kappa_2 y + \kappa_4)h_1],\\
&A_{33}(x, y) = (\kappa_3 x + \kappa_4)(y + a_6)h_6,\\
&A_{32}(x, y) = [(x + a_6)(\kappa_1 y + \kappa_2)h_6 + a_2 y(x + a_5)(\kappa_3 y + \kappa_4)h_2],\\
&A_{22}(x, y) = \kappa_2 a_1 h_1 x^2 y^2 + [(\kappa_4 a_1 h_1 + \kappa_1 a_2 h_2)xy(x + y)]\\
&\qquad + \kappa_3 a_2 h_2(x^2 + y^2) + [(\kappa_4 a_2 h_2 + \kappa_1 h_6)(x + y)] + \kappa_2 h_6,\\
&A_{31}(x, y) = A_{13}(y, x), \qquad A_{21}(x, y) = A_{12}(y, x), \qquad A_{23}(x, y) = A_{32}(y, x).
\end{aligned} \tag{A.1}$$

Case 3.2. In case 3.2 with $\nu = 0$, $m = n_1 = n_2$,

$$
\begin{aligned}
A_{11}(x,y) &= [\kappa_1 xy + \lambda_1(x+y+1)][(h_1 - n_1)xy + n_1(x+1)(y+1)],\\
A_{33}(x,y) &= [\lambda_1(xy+x+y) + \kappa_1][(h_6 - n_1) + n_1(x+1)(y+1)],\\
A_{13}(x,y) &= A_{31}(y,x) = [\kappa_1(xy+y+1) + \lambda_1 x][(h_2 - n_1)x + n_1(x+1)(y+1)],\\
A_{12}(x,y) &= A_{21}(y,x) = [\kappa_1(xy+y+1) + \lambda_1 x][(h_1 - n_1)xy + n_1(x+1)(y+1)]\\
&\quad + [\kappa_1 xy + \lambda_1(x+y+1)][(h_2 - n_1)x + n_1(x+1)(y+1)],\\
A_{32}(x,y) &= A_{23}(y,x) = [\kappa_1 + \lambda_1(xy+y+x)][(h_2 - n_1)y + n_1(x+1)(y+1)]\\
&\quad + [\kappa_1(xy+y+1) + \lambda_1 y][(h_6 - n_1) + n_1(x+1)(y+1)],\\
A_{22}(x,y) &= [\kappa_1 h_1 + 2\kappa_1 n_1 + \lambda_1 n_1]x^2y^2 + [\lambda_1 h_2 + 2\lambda_1 n_1 + \kappa_1 n_1](x^2+y^2)\\
&\quad + [\kappa_1 h_1 + \lambda_1 h_2 + 3\kappa_1 n_1 + 3\lambda_1 n_1]xy(x+y)\\
&\quad + [\kappa_1 h_6 + \lambda_1 h_2 + 3\kappa_1 n_1 + 3\lambda_1 n_1](x+y) + [\kappa_1 h_6 + 2\kappa_1 n_1 + \lambda_1 n_1].
\end{aligned}
\tag{A.2}
$$

Case 3.3. In case 3.3 with $\lambda_5 = \lambda_1$, $\kappa_7 = \kappa_2$, $\nu = 1$,

$$
\begin{aligned}
A_{11}(x,y) &= \lambda_1 x^{-1}y^{-1}(x+y)[h_1 xy + n_1(x+y)],\\
A_{33}(x,y) &= \lambda_1 xy(x+y)[h_6 + n_1 x + n_1 y],\\
A_{13}(x,y) &= A_{31}(y,x) = (\lambda_1 x + \kappa_2 y)[h_2 x + n_1 y]xy^{-1},\\
A_{32}(x,y) &= A_{23}(y,x) = \lambda_1 y(x+y)(h_2 y + n_1 x) + (\lambda_1 y + \kappa_2 x)y(h_6 + n_1 x + n_1 y),\\
A_{12}(x,y) &= A_{21}(y,x) = (\lambda_1 x + \kappa_2 y)[h_1 xy + n_1(x+y)]y^{-1} + \lambda_1(x+y)[h_2 x + n_1 y]y^{-1},\\
A_{22}(x,y) &= \kappa_2 h_1 xy(x+y) + [\lambda_1 h_2 + 2\lambda_1 n_1 + \kappa_1 n_1](x^2+y^2) + \kappa_2 h_6(x+y).
\end{aligned}
\tag{A.3}
$$

Case 3.4. In case 3.4 with $L_5 = n_1$ and $\nu = -1$, we have

$$
\begin{aligned}
A_{11}(x,y) &= xy(\kappa_1 xy + \lambda_3)(h_1 xy + n_1),\\
A_{13}(x,y) &= A_{31}(y,x) = \frac{\kappa_1(xy+1)y(h_2 x + n_1 + n_1 xy)}{x},\\
A_{33}(x,y) &= \frac{(\lambda_3 xy + \kappa_1)(h_6 + n_1 xy)}{xy},\\
A_{12}(x,y) &= A_{21}(y,x) = \kappa_1(xy+1)y(h_1 xy + n_1) + xy(\kappa_1 xy + \lambda_3)(h_2 + n_1 y + n_1 x^{-1}),\\
A_{32}(x,y) &= A_{23}(y,x) = (\lambda_3 xy + \kappa_1)(h_2 + n_1 y^{-1} + n_1 x) + \kappa_1(xy+1)(n_1 x + h_6 y^{-1}),\\
A_{22}(x,y) &= [\kappa_1(2n_1 + h_1) + \lambda_3 n_1]x^2y^2 + \lambda_3 h_2(x+y)(xy+1) + [\kappa_1(2n_1 + h_6) + \lambda_3 n_1].
\end{aligned}
\tag{A.4}
$$

Case 3.5. In case 3.5 with $\lambda_7 = \lambda_2$, $m_1 = n_2 = 0$, $\nu = 0$, we have

$$
\begin{aligned}
&A_{11}(x,y) = \lambda_3(h_1 xy + n_1), \qquad A_{13}(x,y) = A_{31}(y,x) = \lambda_2 y(n_1 y + h_2 x),\\
&A_{22}(x,y) = \lambda_2 h_1 x^2y^2 + \lambda_3 h_2(x^2+y^2) + \lambda_2 h_6, \qquad A_{33}(x,y) = \lambda_3 xy(n_1 xy + h_6),\\
&A_{12}(x,y) = A_{21}(y,x) = \lambda_2 y(h_1 xy + n_1) + \lambda_3(h_2 x + n_1 y),\\
&A_{32}(x,y) = A_{23}(y,x) = \lambda_3 y(h_2 xy + n_1 x^2) + \lambda_2(h_6 x + n_1 x^2 y).
\end{aligned}
\tag{A.5}
$$

Case 3.6. In case 3.6 with $\lambda_7 = \lambda_2$, $m_1 = n_1 = n_2 = 0$, $\nu = -1$, we have

$$
\begin{aligned}
&A_{11}(x,y) = \lambda_3 h_1 x^2y^2, \qquad A_{33}(x,y) = \lambda_3 h_6, \qquad A_{12}(x,y) = y(\lambda_2 h_1 x^2y^2 + \lambda_3 h_2),\\
&A_{13}(x,y) = \lambda_2 y^2 h_2, \qquad A_{32}(x,y) = \lambda_3 x^2 y h_2 + \lambda_2 y^{-1} h_6, \qquad A_{22}(x,y) = 0,\\
&A_{21}(x,y) = A_{12}(y,x), \qquad A_{23}(x,y) = A_{32}(y,x), \qquad A_{31}(x,y) = A_{13}(y,x).
\end{aligned}
\tag{A.6}
$$

Case 3.7. In case 3.7 with $\nu = -2$, we have

$$\begin{aligned} &A_{11}(x,y) = x^3y^3[\kappa_1 xy + \lambda_3]h_1, \qquad A_{12}(x,y) = xy^2[\kappa_1 h_1 xy + \kappa_1 h_2 xy + \lambda_3 h_2],\\ &A_{22}(x,y) = \lambda_3[x^2y^2+1]h_2, \qquad A_{13}(x,y) = \kappa_1 h_2 y^2,\\ &A_{32}(x,y) = \lambda_3 h_2 x + \kappa_1 y^{-1}(h_2+h_6), \qquad A_{33}(x,y) = (\kappa_1 x^{-2}y^{-2} + \lambda_3 x^{-1}y^{-1})h_6,\\ &A_{21}(x,y) = A_{12}(y,x), \qquad A_{31}(x,y) = A_{13}(y,x), \qquad A_{23}(x,y) = A_{32}(y,x). \end{aligned} \tag{A.7}$$

Case 4.1. In case 4.1 with $\nu = 0$, $f_2(y) = f_7(y) = \kappa_2 y + \lambda_2$, we have

$$\begin{aligned} &A_{11}(x,y) = xy[\kappa_1 xy + \lambda_1(x+y) + \lambda_3]h_1, \qquad A_{12}(x,y) = xy[\kappa_2 xy + \lambda_2 y + \lambda_3 x + \lambda_4]h_1,\\ &A_{22}(x,y) = xy[\lambda_2 xy + \lambda_4 x + \lambda_4 y]h_1 + [\kappa_1 x + \kappa_1 y + \kappa_2]h_6,\\ &A_{32}(x,y) = [k_1 xy + \lambda_1 y + k_2 x + \lambda_2]h_6, \qquad A_{33}(x,y) = [\lambda_1 xy + \lambda_3 x + \lambda_3 y + \lambda_4]h_6,\\ &A_{21}(x,y) = A_{12}(y,x), \qquad A_{23}(x,y) = A_{32}(y,x), \qquad A_{13}(x,y) = A_{31}(x,y) = 0. \end{aligned} \tag{A.8}$$

Case 4.2. In case 4.2 with $\nu = -1$, $f_2(y) = f_7(y) = \lambda_2$, we have

$$\begin{aligned} &A_{11}(x,y) = x^2y^2(\kappa_1 xy + \lambda_3)h_1, \qquad A_{12}(x,y) = xy^2(\lambda_2 xy + \lambda_4)h_1,\\ &A_{22}(x,y) = \lambda_4 h_1 x^2y^2 + \kappa_1 h_6, \qquad A_{32}(x,y) = (\kappa_1 x + \lambda_2 y^{-1})h_6,\\ &A_{33}(x,y) = (\lambda_3 + \lambda_4 x^{-1}y^{-1})h_6,\\ &A_{13}(x,y) = A_{31}(x,y) = 0, \qquad A_{21}(x,y) = A_{12}(y,x), \qquad A_{23}(x,y) = A_{32}(y,x). \end{aligned} \tag{A.9}$$

Case 4.3. In case 4.3 with $\nu = -2$, $f_2(y) = f_1(y) = 0$, we have

$$\begin{aligned} &A_{11}(x,y) = x^3y^3(\kappa_1 xy + \lambda_3)h_1, \qquad A_{12}(x,y) = \lambda_4 x^2y^3 h_1,\\ &A_{32}(x,y) = \kappa_1 y^{-1}h_6, \qquad A_{33}(x,y) = (\lambda_3 + \lambda_4 x^{-1}y^{-1})x^{-1}y^{-1}h_6,\\ &A_{21}(x,y) = A_{12}(y,x), \qquad A_{23}(x,y) = A_{32}(y,x),\\ &A_{22}(x,y) = A_{13}(x,y) = A_{31}(x,y) = 0. \end{aligned} \tag{A.10}$$

Case 4.4. In case 4.4 with $h_1 = h_6$, $\nu = 1$, we have

$$\begin{aligned} &A_{11}(x,y) = \lambda_3 h_1, \quad A_{33}(x,y) = \lambda_3 h_6 x^2y^2, \quad A_{13}(x,y) = \lambda_7 h_2 x^2, \quad A_{31}(x,y) = \lambda_2 h_2 y^2,\\ &A_{12}(x,y) = \lambda_7 h_1 y + \lambda_3 h_2 x^2 y^{-1}, \qquad A_{21}(x,y) = \lambda_2 h_1 x + \lambda_3 h_2 y^2 x^{-1},\\ &A_{32}(x,y) = \lambda_3 h_2 y^3 + \lambda_2 h_6 x^2 y, \qquad A_{23}(x,y) = \lambda_3 h_2 x^3 + \lambda_7 h_6 xy^2, \qquad A_{22}(x,y) = 0. \end{aligned} \tag{A.11}$$

Case 4.5. In case 4.5 with $h_1 = h_6$, $\nu = 2$, we have

$$\begin{aligned} &A_{11}(x,y) = \kappa_1 h_1, \qquad A_{33}(x,y) = \kappa_1 h_6 x^2y^2, \qquad A_{13}(x,y) = (\kappa_7 x^3y^{-1} + \kappa_1 x^4 y^{-2})h_2,\\ &A_{31}(x,y) = (\kappa_2 y^3x^{-1} + \kappa_1 y^4x^{-2})h_2, \qquad A_{22}(x,y) = \kappa_7 y^2 h_1 + \kappa_2 x^2 h_6,\\ &A_{12}(x,y) = \kappa_7 x h_1 + \kappa_1 x^2 y^{-1}(h_1+h_2), \qquad A_{21}(x,y) = \kappa_2 y h_1 + \kappa_1 y^2 x^{-1}(h_1+h_2),\\ &A_{32}(x,y) = \kappa_2 xy^2 h_6 + \kappa_1 y^3(h_2+h_6), \qquad A_{23}(x,y) = \kappa_7 x^2 y h_6 + \kappa_1 x^3(h_2+h_6), \end{aligned} \tag{A.12}$$

Case 4.6. In case 4.6 with $\nu = -1$, we have

$$\begin{aligned} &A_{11}(x,y) = x^2y^2(\kappa_1 xy + \lambda_3)h_1, \qquad A_{12}(x,y) = A_{21}(y,x) = \kappa_1 x^2y^2(h_1+h_2) + \lambda_3 xyh_2,\\ &A_{22}(x,y) = \lambda_3 xy(x+y)h_2, \qquad A_{13}(x,y) = A_{31}(y,x) = k_1 xyh_2,\\ &A_{32}(x,y) = A_{23}(y,x) = (\lambda_3 y + \kappa_1)h_2, \qquad A_{33}(x,y) = 0. \end{aligned} \tag{A.13}$$

Appendix B.

In this appendix, we present the expressions for $A_{13}(x, y)z^2 + A_{23}(x, y)z + A_{33}(x, y)$ and $A_{11}(y, z)x^2 + A_{12}(y, z)x + A_{13}(y, z)$ in cases 3.1–3.7 which are used to obtain the dual G-type of mappings in cases 5.1–5.7 with the use of (1.12).

Case 3.1.

$$\begin{aligned}A_{13}(x, y)z^2 + A_{23}(x, y)z + A_{33}(x, y) &= [\kappa_1 xz + \kappa_2 z + \kappa_3 x + \kappa_4][a_2 xz(y + a_5)h_2 + (y + a_6)h_6],\\ A_{11}(y, z)x^2 + A_{12}(y, z)x + A_{13}(y, z) &= [\kappa_1 xz + \kappa_2 z + \kappa_3 x + \kappa_4][xyz(y + a_1)h_1 + y(y + a_2)h_2].\end{aligned} \tag{B.1}$$

Case 3.2.

$$\begin{aligned}A_{13}(x, y)z^2 + A_{23}(x, y)z + A_{33}(x, y) &= \alpha(x, y, z)[(h_2 - n_1)xz + (h_6 - n_1)\\ &\quad + n_1(x + 1)(y + 1)(z + 1)],\\ A_{11}(y, z)x^2 + A_{12}(y, z)x + A_{13}(y, z) &= \alpha(x, y, z)[(h_1 - n_1)xyz + (h_2 - n_1)y\\ &\quad + n_1(x + 1)(y + 1)(z + 1)],\\ \alpha(x, y, z) &= [\kappa_1(xyz + yz + z + 1) + \lambda_1(x + y + xy + xz)].\end{aligned} \tag{B.2}$$

Case 3.3.

$$\begin{aligned}A_{13}(x, y)z^2 + A_{23}(x, y)z + A_{33}(x, y) &= \beta(x, y, z)[n_1 xz + h_2 x^2 z y^{-1} + h_6 x + n_1 x^2 + n_1 xy],\\ A_{11}(y, z)x^2 + A_{12}(y, z)x + A_{13}(y, z) &= \beta(x, y, z)[h_1 x + n_1 xz^{-1} + n_1 xy^{-1} + n_1 + h_2 yz^{-1}],\\ \beta(x, y, z) &= [\lambda_1 xz + \kappa_2 yz + \lambda_1 y^2 + \lambda_1 xy].\end{aligned} \tag{B.3}$$

Case 3.4.

$$\begin{aligned}A_{13}(x, y)z^2 + A_{23}(x, y)z + A_{33}(x, y) &= \delta(x, y, z)[h_2 yz + n_1 y^2 z + n_1 x^{-1} yz\\ &\quad + n_1 y + h_6 x^{-1}]x^{-1} y^{-1} z^{-1},\\ A_{11}(y, z)x^2 + A_{12}(y, z)x + A_{13}(y, z) &= \delta(x, y, z)[h_1 yz + n_1 x^{-1} z + h_2 x^{-1} + n_1 x^{-1} y^{-1} + n_1],\\ \delta(x, y, z) &= [\kappa_1 xyz + \kappa_1 z + \lambda_3 x + \kappa_1 y^{-1}].\end{aligned} \tag{B.4}$$

Case 3.5.

$$\begin{aligned}A_{13}(x, y)z^2 + A_{23}(x, y)z + A_{33}(x, y) &= [\lambda_3 x + \lambda_2 z][h_2 xyz + n_1 y^2(x + z) + h_6 y],\\ A_{11}(y, z)x^2 + A_{12}(y, z)x + A_{13}(y, z) &= [\lambda_3 x + \lambda_2 z][h_1 yxz + n_1(x + z) + h_2 y].\end{aligned} \tag{B.5}$$

Case 3.6.

$$\begin{aligned}A_{13}(x, y)z^2 + A_{23}(x, y)z + A_{33}(x, y) &= [\lambda_3 xy + \lambda_2 yz][h_2 yz + h_6 x^{-1} y^{-1}],\\ A_{11}(y, z)x^2 + A_{12}(y, z)x + A_{13}(y, z) &= [\lambda_3 xy + \lambda_2 yz][h_1 xyz^2 + h_2 y^{-1} z].\end{aligned} \tag{B.6}$$

Case 3.7.

$$\begin{aligned}A_{13}(x, y)z^2 + A_{23}(x, y)z + A_{33}(x, y) &= [\kappa_1 xyz + \lambda_3 x + \kappa_1 y^{-1}][h_2 x^{-1} yz + h_6 x^{-2} y^{-1}],\\ A_{11}(y, z)x^2 + A_{12}(y, z)x + A_{13}(y, z) &= [\kappa_1 xyz + \lambda_3 x + \kappa_1 y^{-1}][h_1 xy^3 z^3 + h_2 yz^2].\end{aligned} \tag{B.7}$$

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